

# Updating the Fundamental Theorem of Homomorphism of General Universal Algebras

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## Abstract

From the fundamental theorem of homomorphisms, it is well known that any homomorphism of groups (or rings, or modules, or vector spaces and of general universal algebras) can be decomposed as a composition of a monomorphism and an epimorphism. This paper provides the uniqueness of such decomposition up to the level of associates in the case of general universal algebras

**Keywords:** Algebra of a given type; Homomorphisms; Kernel of a homomorphism; Congruence Relations and an associate of a homomorphism.

## 1. Introduction

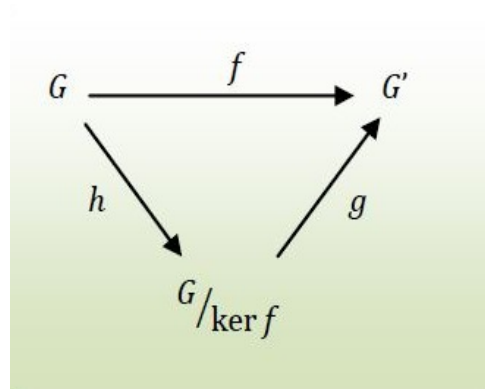
It is well known that, if  $f: G \rightarrow G'$  is a homomorphism of groups (or rings or modules) the quotient group  $G/\ker f$  is isomorphic to the image of  $f$  which is a subgroup of the codomain group  $G'$ . This isomorphism  $g$  is simply induced by  $f$ , in the sense that,  $g$  can be defined by  $g(a + \ker f) = f(a)$  for any coset  $a + \ker f$ ,  $a \in G$ . Also, we have the natural epimorphism  $h: G \rightarrow G/\ker f$ , defined by  $h(a) = a + \ker f$  for all  $a \in G$ . In other words, we have a decomposition  $f = g \circ h$ ; where  $g$  is a monomorphism and  $h$  is an epimorphism. This is known as the Fundamental Theorem of Homomorphisms [3, 11]. This result can be extended to any homomorphism  $f: A \rightarrow A'$  of universal algebras of the same type by considering the binary relation  $\theta$  on the domain algebra  $A$  defined by,

$$\theta = \{(a, b) \in A \times A : f(a) = f(b)\}$$

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**Figure1:** Decomposition of homomorphism of groups

and the quotient algebra  $A/\theta$ . In the general case of universal algebras, this  $\theta$  is defined as the kernel of  $f$  and is actually a congruence relation on the domain algebra  $A$  and  $A/\theta$  is the set of all congruence classes of elements of  $A$  corresponding to  $\theta$  [8]. In the familiar cases of homomorphisms of groups (or rings or modules) the kernel of  $f$  is a normal subgroup (or an ideal or a submodule respectively) of the domain and the congruence classes are precisely the cosets of the conventional kernels [5, 9, 10]. In all these cases there is an order isomorphism of the set of normal subgroups(or ideals or submodules) onto the set of congruence relations on the domain algebra. In this scenario, we can say that any homomorphism  $f$  of algebras of any type can be decomposed as a composition of a monomorphism and an epimorphism.

It is natural to question ourselves that, is this decomposition done uniquely in one way? Of course not in one and only one way and this paper provides the uniqueness of such a decomposition of homomorphisms upto the level of associate in the case of homomorphisms of general universal algebras.

## 2. Method/Approach

Let  $A$  and  $B$  be algebras of a given type  $\mathcal{F}$  and let  $f: A \rightarrow B$ , be any homomorphism. Let us consider the binary relation  $K_f$  defined by:

$$K_f = \{(x, y) \in A \times A : f(x) = f(y)\}$$

Then  $K_f$  becomes a congruence relation on  $A$  [6, 11]. Also, the natural map  $h: A \rightarrow A/K_f$ , defined by  $h(a) = K_f(a)$ , the congruence class of  $a$  corresponding to  $K_f$ , is an epimorphism and the function  $g: A/K_f \rightarrow B$ , defined by  $g(K_f(a)) = f(a)$  for all  $a \in A$ , is a monomorphism. Now, we have the decomposition  $f = g \circ h$ .

To discuss about the uniqueness of the monomorphism  $g$  and the epimorphism  $h$  in this decomposition, we first define the notion of an associate of a homomorphism of algebras and then we prove the uniqueness of such a decomposition upto the level of associates; that is, if  $f$  can be decomposed in two ways as  $f = g \circ h$  and  $f = g' \circ h'$ , where  $g$  and  $g'$  are monomorphisms and  $h$  and  $h'$  are epimorphisms then we will prove that

$g \sim g'$  and  $h \sim h'$ . This says that, the decomposition of a homomorphism  $f$  as a composition of a monomorphism and an epimorphism is unique up to associate. In this vein we generalize and unify all the fundamental theorems of homomorphisms and isomorphisms.

### 3. Preliminaries

#### 3.1. Definition of Algebras

**Definition 3.1.1.** For a nonempty set  $A$  and a nonnegative integer  $n$ , we define  $A^0 = \{\emptyset\}$ , and for  $n > 0$ ,  $A^n$  is the set of  $n$ -tuples of elements from  $A$ . An  $n$ -ary operation (or function) on  $A$  is any function  $f$  from  $A^n$  to  $A$ ;  $n$  is the *arity* (or *rank*) of  $f$ . A *finitary* operation is an  $n$ -ary operation, for some  $n$ . The image of  $(a_1, a_2, \dots, a_n)$  under an  $n$ -ary operation  $f$  is denoted by  $f(a_1, a_2, \dots, a_n)$ . An operation  $f$  on  $A$  is called a *nullary* operation (or *constant*) if its arity is zero; it is completely determined by the image  $f(\emptyset)$  in  $A$  of the only element  $\emptyset$  in  $A^0$ , and as such it is convenient to identify it with the element  $f(\emptyset)$ . Thus a nullary operation is thought of as an element of  $A$ . An operation  $f$  on  $A$  is *unary*, *binary*, or *ternary* if its arity is 1, 2, or 3, respectively. [1]

**Definition 3.1.2.** A *type* (or *language*) of algebras is a set  $\mathcal{F}$  of *function symbols* such that a nonnegative integer  $n$  is assigned to each member  $f$  of  $\mathcal{F}$ . This integer is called the *arity* (or *rank*) of  $f$ , and  $f$  is said to be an  $n$ -ary operation symbol. The subset of  $n$ -ary function symbols in  $\mathcal{F}$  is denoted by  $\mathcal{F}_n$ . [1]

**Definition 3.1.3.** If  $\mathcal{F}$  is a type of algebras then an *algebra*  $A$  of type  $\mathcal{F}$  is an ordered pair  $\langle A, F \rangle$  where  $A$  is a nonempty set and  $F$  is a family of finitary operations on  $A$  indexed by the language  $\mathcal{F}$  such that corresponding to each  $n$ -ary function symbol  $f$  in  $\mathcal{F}$  there is an  $n$ -ary operation  $f^A$  on  $A$ . The set  $A$  is called the *universe* (or *underlying set*) of  $\langle A, F \rangle$ , and the  $f^A$ 's are called the *fundamental operations* of  $A$ . [1]

If  $F$  is finite, say  $\mathcal{F} = \{f_1, f_2, \dots, f_k\}$ , we often write  $\langle A, f_1, f_2, \dots, f_k \rangle$  for  $\langle A, F \rangle$ , usually adopting the convention:

$$\text{arity } f_1 \geq \text{arity } f_2 \geq \dots \geq \text{arity } f_k.$$

#### 3.2. Homomorphisms of Algebras

**Definition 3.2.1:** Let  $A$  and  $B$  be algebras of a given type  $\mathcal{F}$ . A mapping  $\alpha: A \rightarrow B$  is called a homomorphism, if the following are satisfied. [1, 4 & 7]

- i. If  $f \in \mathcal{F}$  is a nullary operation symbol, then  $\alpha(f^A) = f^B$
- ii. If  $f \in \mathcal{F}$  is an  $n$ -ary operation symbol,  $n > 0$  and  $a_1, a_2, \dots, a_n \in A$ , then

$$\alpha(f^A(a_1, a_2, \dots, a_n)) = f^B(\alpha(a_1), \alpha(a_2), \dots, \alpha(a_n))$$

**Theorem 3.2.1:** Let  $A$ ,  $B$  and  $C$  be algebras of a given type  $\mathcal{F}$ . Let  $\alpha: A \rightarrow B$  and  $\beta: B \rightarrow C$  be homomorphisms. Then  $\beta \circ \alpha: A \rightarrow C$  is also a homomorphism [7].

**Proof:** Let  $f \in \mathcal{F}$  be a nullary operation symbol. Then

$$\begin{aligned}\beta \circ \alpha(f^A) &= \beta(\alpha(f^A)) \\ &= \beta(f^B) && (\because \alpha \text{ is a homomorphism}) \\ &= f^C && (\because \beta \text{ is a homomorphism})\end{aligned}$$

Let  $f \in \mathcal{F}$  be an  $n$ -ary operation symbol,  $n > 0$  and  $a_1, a_2, \dots, a_n \in A$ . Then,

$$\begin{aligned}\beta \circ \alpha(f^A(a_1, a_2, \dots, a_n)) &= \beta(\alpha(f^A(a_1, a_2, \dots, a_n))) \\ &= \beta(f^B(\alpha(a_1), \alpha(a_2), \dots, \alpha(a_n))) && (\because \alpha \text{ is a homomorphism}) \\ &= f^C(\beta(\alpha(a_1)), \beta(\alpha(a_2)), \dots, \beta(\alpha(a_n))) \\ &&& (\because \beta \text{ is a homomorphism}) \\ &= f^C(\beta \circ \alpha(a_1), \beta \circ \alpha(a_2), \dots, \beta \circ \alpha(a_n))\end{aligned}$$

Therefore,  $\beta \circ \alpha: A \rightarrow C$  is also a homomorphism.

**Definition 3.2.2:** Let  $A$  and  $B$  be algebras of a given type  $\mathcal{F}$  and  $\alpha: A \rightarrow B$  be any function. Then

- i.  $\alpha$  is called a monomorphism, if it is an injective homomorphism.
- ii.  $\alpha$  is called an epimorphism, if it is a surjective homomorphism.
- iii.  $\alpha$  is called an isomorphism, if it is a bijective homomorphism. [1, 6 & 7]

### 3.3. Congruence Relations

**Definition 3.3.1:** Let  $A$  be an algebra of type  $\mathcal{F}$  and  $\theta$  be an equivalence relation on  $A$ . Then  $\theta$  is said to be a congruence relation on  $A$ , if the following is satisfied for any  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in A$ ,

If  $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n) \in \theta$ ,  $f \in \mathcal{F}$  is an  $n$ -ary operation symbol and  $n > 0$ , then  $(f^A(a_1, a_2, \dots, a_n), f^A(b_1, b_2, \dots, b_n)) \in \theta$ . [1, 2, 6 & 7]

In other words, a congruence relation is an equivalence relation on an algebraic structure (such as groups, rings or vector spaces and of general universal algebras) that is compatible with the structure.

In the next two theorems we will observe that kernels of homomorphisms and congruence relations are same.

**Theorem 3.3.1:** Let  $A$  and  $B$  be algebras of a given type  $\mathcal{F}$ . Let  $\alpha: A \rightarrow B$  be a homomorphism. Then kernel of  $\alpha$  defined by;  $\ker \alpha = \{(a, b) \in A \times A : \alpha(a) = \alpha(b)\}$  is a congruence relation on  $A$ . [1, 2, 6]

**Proof:** Put  $\theta = \ker \alpha$ . Then it is clear that,  $\theta$  is an equivalence relation on  $A$ .

Now, for any  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in A$ , let  $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n) \in \theta$  and let  $f \in \mathcal{F}$  be an  $n$ -ary operation symbol and  $n > 0$ , then  $\alpha(a_i) = \alpha(b_i)$  for all  $1 \leq i \leq n$ .

Therefore,

$$\begin{aligned} \alpha(f^A(a_1, a_2, \dots, a_n)) &= f^B(\alpha(a_1), \alpha(a_2), \dots, \alpha(a_n)) & (\because \alpha \text{ is a homomorphism}) \\ &= f^B(\alpha(b_1), \alpha(b_2), \dots, \alpha(b_n)) \\ &= \alpha(f^A(b_1, b_2, \dots, b_n)) & (\because \alpha \text{ is a homomorphism}) \end{aligned}$$

Thus,  $(f^A(a_1, a_2, \dots, a_n), f^A(b_1, b_2, \dots, b_n)) \in \ker \alpha = \theta$  and hence,  $\ker \alpha$  is a congruence relation on  $A$ .

**Theorem 3.3.2:** Let  $A$  be an algebra of type  $\mathcal{F}$  and  $\theta$  be a congruence relation on  $A$ . Then there exists an algebra  $B$  of type  $\mathcal{F}$  and a homomorphism  $\alpha: A \rightarrow B$  such that  $\theta = \ker \alpha$ . [2]

**Proof:** Put  $\theta = \ker \alpha = \{(a, b) \in A \times A : \alpha(a) = \alpha(b)\}$  and let  $A/\theta = \{\theta(a) : a \in A\}$  be the quotient of  $A$  over  $\theta$ , where  $\theta(a) = \{b \in A : (a, b) \in \theta\}$ . Then  $A/\theta$  is an algebra of the given type  $\mathcal{F}$ .

Now consider a natural epimorphism  $\alpha: A \rightarrow A/\theta$  defined by  $\alpha(a) = \theta(a)$  for all  $a \in A$ .

Then,

$$\begin{aligned} \ker \alpha &= \{(a, b) \in A \times A : \alpha(a) = \alpha(b)\} \\ &= \{(a, b) \in A \times A : \theta(a) = \theta(b)\} \\ &= \{(a, b) \in A \times A : (a, b) \in \theta\} \\ &= \theta \end{aligned}$$

Therefore, the required algebra and a homomorphism are  $A/\theta$  and  $\alpha$  such that  $\ker \alpha = \theta$ . ■

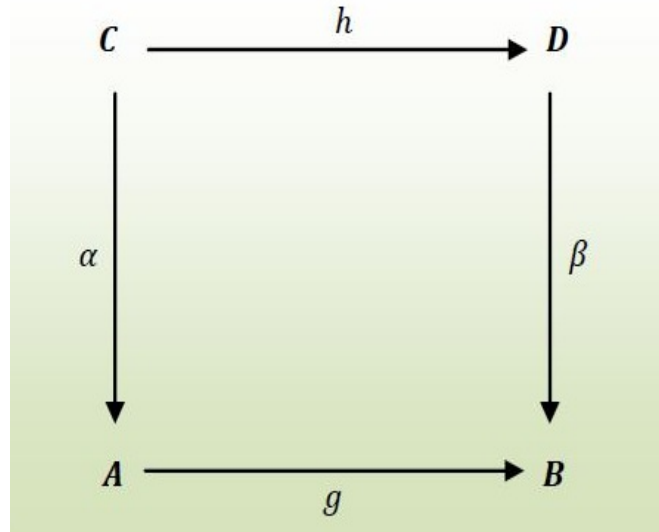
Now, from the above two theorems we observe that, kernels of homomorphisms from an algebra  $A$  and congruence relations on  $A$  are the same.

#### 4. The Updated Fundamental Theorem of Homomorphisms of General Universal Algebras

##### 4.1. An associate of a Homomorphism of General Universal Algebras

**Definition 4.1.1:** Let  $A, B, C$  and  $D$  be algebras of a given type  $\mathcal{F}$ . Let  $g: A \rightarrow B$  and  $h: C \rightarrow D$  be

homomorphisms. Then  $g$  is said to be an associate of  $h$ , if there exist two isomorphisms  $\alpha: C \rightarrow A$  and  $\beta: D \rightarrow B$  such that, the diagram



**Figure 2:** An associate of a homomorphism

is commutative; in the sense that,  $g \circ \alpha = \beta \circ h$ .

**Remark:** Following the above definition we can observe that the binary relation “ $\sim$ ” forms an equivalence relation on the class of all algebras of a given type. Furthermore, if any two homomorphisms are given to be an associate to each other, then we can draw the properties of a homomorphism like injectivity, surjectivity and bijectivity which is held in one of the two associate homomorphisms to the other.

#### 4.2. The Updated Fundamental Theorem of Homomorphisms of Algebras

**Theorem 4.2.1(The Updated Fundamental Theorem of Homomorphisms of Algebras):** Let  $A$  and  $B$  be algebras of a given type  $\mathcal{F}$  and  $\pi: A \rightarrow B$  be a homomorphism. Then,

1. There exists an algebra  $X$  of the given type  $\mathcal{F}$ , an epimorphism  $h: A \rightarrow X$  and a monomorphism  $g: X \rightarrow B$  such that;

$$\pi = g \circ h \quad (\text{i.e. } \pi(a) = g(h(a)) \text{ for all } a \in A)$$

This property is known as decomposition of homomorphisms and

2. This decomposition is unique upto associate; in the sense that, if  $\pi = g \circ h$  and  $\pi = g' \circ h'$  are two decompositions of  $\pi$  where  $X$  and  $Y$  are algebras of the given type  $\mathcal{F}$ ,  $h: A \rightarrow X$  and  $h': A \rightarrow Y$  are epimorphisms and  $g: X \rightarrow B$  and  $g': Y \rightarrow B$  are monomorphisms, then  $g \sim g'$  and  $h \sim h'$ .

**Proof:**

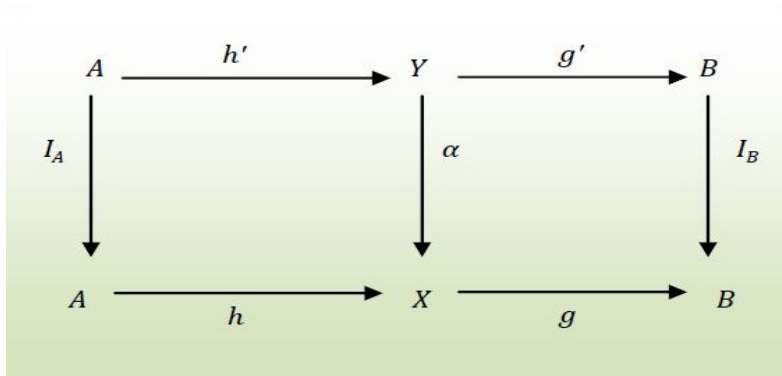
1. Put  $\theta = \ker \pi = \{(a, b) \in A \times A : \pi(a) = \pi(b)\}$ . Then it is clear that  $A/\theta = \{\theta(a) : a \in A\}$ , the quotient of  $A$  over  $\theta$ , is an algebra of the given type  $\mathcal{F}$ . Now consider a natural epimorphism  $\alpha: A \rightarrow A/\theta$  defined by  $\alpha(a) = \theta(a)$  for all  $a \in A$  and a monomorphism  $g: A/\theta \rightarrow B$  defined by  $g(\theta(a)) = \pi(a)$  for all  $\theta \in A/\theta, a \in A$ . Then, for any  $a \in A$  consider,

$$\begin{aligned} g \circ h(a) &= g(h(a)) \\ &= g(\theta(a)) && \text{(by the definition of } h) \\ &= \pi(a) && \text{(by the definition of } g) \end{aligned}$$

Thus,  $g \circ h = \pi$  is the required decomposition of  $\pi$ .

2. Now we prove the uniqueness (upto associate) of the above decomposition. Suppose that  $\pi = g \circ h$  and  $\pi = g' \circ h'$  are two decompositions of  $\pi$  where  $X$  and  $Y$  are algebras of the given type  $\mathcal{F}$ ,  $h: A \rightarrow X$  and  $h': A \rightarrow Y$  are epimorphisms and  $g: X \rightarrow B$  and  $g': Y \rightarrow B$  are monomorphisms.

**Claim 1:-**  $h \sim h'$  ( $h$  and  $h'$  are associate to each other) and  $g \sim g'$



**Figure 3:** Two decompositions of  $\pi$

Now consider the identity map  $I_A$  on an algebra  $A$ , which is an isomorphism. On the other hand since  $h': A \rightarrow Y$  is an epimorphism, every element of  $Y$  can be expressed as  $h'(a)$  for some  $a \in A$ . Define  $\alpha: Y \rightarrow X$  by:

$$\alpha(h'(a)) = h(a) \text{ for all } h'(a) \in Y \text{ where } a \in A.$$

Then we prove that:

a)  $\alpha$  is well defined; for,

For any  $y_1, y_2 \in Y$  there exists  $a_1$  and  $a_2 \in A$  such that  $y_1 = h'(a_1)$  and  $y_2 = h'(a_2)$

Therefore,

$$\begin{aligned}
 y_1 = y_2 &\Rightarrow h'(a_1) = h'(a_2) \text{ in } Y \\
 &\Rightarrow g'(h'(a_1)) = g'(h'(a_2)) \text{ in } B \\
 &\Rightarrow g' \circ h'(a_1) = g' \circ h'(a_2) \text{ in } B \\
 &\Rightarrow f(a_1) = f(a_2) \text{ in } B \quad (\because f = g' \circ h') \\
 &\Rightarrow g \circ h(a_1) = g \circ h(a_2) \text{ in } B \quad (\because f = g \circ h) \\
 &\Rightarrow g(h(a_1)) = g(h(a_2)) \text{ in } B \\
 &\Rightarrow h(a_1) = h(a_2) \text{ in } X \quad (\because g \text{ is an injection})
 \end{aligned}$$

Therefore,  $\alpha$  is well defined.

b)  $\alpha$  is an injection; for,

Let  $y_1 = h'(a_1)$  and  $y_2 = h'(a_2) \in Y$  for some  $a_1$  and  $a_2 \in A$ . Then

$$\begin{aligned}
 \alpha(y_1) = \alpha(y_2) \text{ in } X &\Rightarrow \alpha(h'(a_1)) = \alpha(h'(a_2)) \text{ in } X \\
 &\Rightarrow h(a_1) = h(a_2) \text{ in } X \quad (\text{by the definition of } \alpha) \\
 &\Rightarrow g(h(a_1)) = g(h(a_2)) \text{ in } B \\
 &\Rightarrow g \circ h(a_1) = g \circ h(a_2) \text{ in } B \\
 &\Rightarrow f(a_1) = f(a_2) \text{ in } B \quad (\because f = g \circ h) \\
 &\Rightarrow g' \circ h'(a_1) = g' \circ h'(a_2) \text{ in } B \quad (\because f = g' \circ h') \\
 &\Rightarrow g'(h'(a_1)) = g'(h'(a_2)) \text{ in } B \\
 &\Rightarrow h'(a_1) = h'(a_2) \text{ in } Y \quad (\because g' \text{ is an injection}) \\
 &\Rightarrow y_1 = y_2 \text{ in } Y
 \end{aligned}$$

Therefore,  $\alpha$  is an injection.



c)  $\alpha$  is a surjection; for,

$$x \in X \Rightarrow x = h(a) \text{ for some } a \in A \quad (\because h \text{ is a surjection})$$

$$\Rightarrow x = \alpha(h'(a)) \text{ for some } a \in A, \text{ where } h'(a) \in Y$$

$$\Rightarrow x = \alpha(y) \text{ for some } y \in Y, \text{ where } y = h'(a)$$

Therefore,  $\alpha$  is a surjection.

d)  $\alpha$  is a homomorphism; for,

Let  $f \in \mathcal{F}$  be a nullary operation symbol. Then  $f^Y \in Y$  and  $f^A \in A$ .

Since  $h'$  is a homomorphism, we have  $h'(f^A) = f^Y$ .

Therefore,

$$\begin{aligned} \alpha(f^Y) &= \alpha(h'(f^A)) \\ &= h(f^A) && (\text{by the definition of } \alpha) \\ &= f^X && (\because h \text{ is a homomorphism}) \end{aligned}$$

Also, Let  $f \in \mathcal{F}$  be an  $n$ -ary operation symbol,  $n > 0$  and  $y_1, y_2, \dots, y_n \in Y$ . Then since  $h'$  is an epimorphism, there exists  $a_1, a_2, \dots, a_n \in A$ , such that  $y_i = h'(a_i)$  for all  $1 \leq i \leq n$  and hence,

$$\begin{aligned} \alpha(f^Y(y_1, y_2, \dots, y_n)) &= \alpha(f^Y(h'(a_1), h'(a_2), \dots, h'(a_n))) \\ &= \alpha(h'(f^A(a_1, a_2, \dots, a_n))) && (\because h' \text{ is a homomorphism}) \\ &= h(f^A(a_1, a_2, \dots, a_n)) \\ &= f^X(h(a_1), h(a_2), \dots, h(a_n)) && (\because h \text{ is a homomorphism}) \\ &= f^X(\alpha(h'(a_1)), \alpha(h'(a_2)), \dots, \alpha(h'(a_n))) \\ &= f^X(\alpha(y_1), \alpha(y_2), \dots, \alpha(y_n)) \end{aligned}$$

Therefore,  $\alpha$  is a homomorphism.

Thus, By the results in (a), (b), (c) and (d) we get that  $\alpha$  is an isomorphism.

Now, for any  $a \in A$ , consider,

$$\alpha h'(a) = \alpha(h'(a))$$

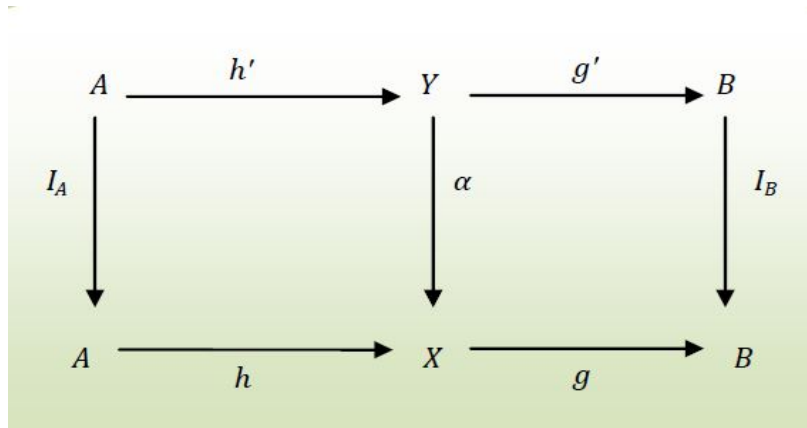
$$= h(a) \quad (\text{by the definition of } \alpha)$$

$$= h(I_A(a)) \quad (\because I_A \text{ is an identity map on } A)$$

$$= h \circ I_A(a)$$

Therefore,  $\alpha h' = h \circ I_A$  and hence  $h \sim h'$  (or  $h$  and  $h'$  are associates).

**Claim 2:-**  $g \sim g'$  (or  $g$  and  $g'$  are associates)



**Figure 4:** Two decompositions of  $\pi$

From claim (1) we have, the identity map  $I_A$  on the domain algebra  $A$  and  $\alpha: Y \rightarrow X$ , which is defined by  $\alpha(h'(a)) = h(a)$  for all  $h'(a) \in Y$  where  $a \in A$ , which are isomorphisms. Also, consider the identity map  $I_B$  on  $B$  which is also an isomorphism.

Since  $h': A \rightarrow Y$  is an epimorphism and hence a surjection, for any  $y \in Y$ , there exist  $a \in A$  such that  $y = h'(a)$ . Then consider,

$$I_B \circ g'(y) = I_B \circ g'(h'(a)) \text{ in } B$$

$$= I_B(g'(h'(a))) \text{ in } B$$

$$= g'(h'(a)) \text{ in } B$$

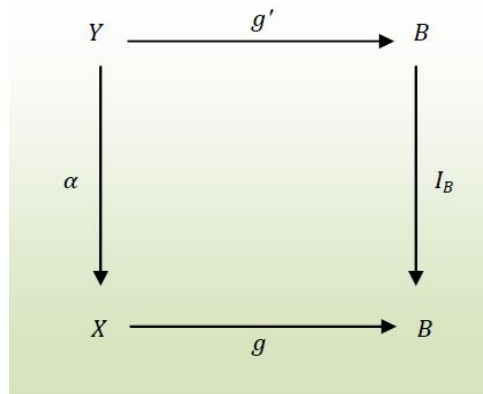
$$= g' \circ h'(a) \text{ in } B$$

$$= f(a) \text{ in } B$$

$$\begin{aligned}
 &= g \circ h(a) \text{ in } B && (\because f = g \circ h) \\
 &= g(h(a)) \text{ in } B \\
 &= g(\alpha(h'(a))) \text{ in } B && (\text{by the definition of } \alpha) \\
 &= g \circ \alpha(h'(a)) \text{ in } B \\
 &= g \circ \alpha(y) \text{ in } B && (\because y = h'(a))
 \end{aligned}$$

Therefore,  $I_B \circ g' = g \circ \alpha$ .

Thus, we have two isomorphisms,  $I_B$  on  $B$  and  $\alpha: Y \rightarrow X$  satisfying that, the diagram;

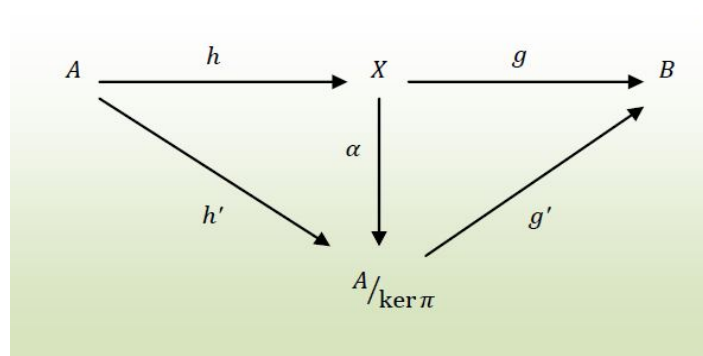


**Figure 5:** Associate homomorphisms  $g$  and  $g'$

is commutative; that is,  $I_B \circ g' = g \circ \alpha$ . Therefore,  $g \sim g'$  (or  $g$  and  $g'$  are associate). ■

**Corollary 4.2.1:** Let  $A$  and  $B$  be algebras of a given type  $\mathcal{F}$  and  $\pi: A \rightarrow B$  be a homomorphism. Let  $\pi = g \circ h$  be any decomposition of  $\pi$  where  $h: A \rightarrow X$  is an epimorphism and  $g: X \rightarrow B$  is a monomorphism for some algebra  $X$  of the given type  $\mathcal{F}$ . Then there exist an isomorphism of  $X$  onto the quotient algebra  $A/\ker \pi$ .

**Proof:** Put  $\theta = \ker \pi = \{(a, b) \in A \times A : \pi(a) = \pi(b)\}$ . Considering the natural epimorphism  $h': A \rightarrow A/\theta$  defined by  $h'(a) = \theta(a)$  for all  $a \in A$  and a monomorphism  $g': A/\theta \rightarrow B$  defined by  $g'(\theta(a)) = f(a)$  for all  $\theta(a) \in A/\theta$ ,  $a \in A$ , we get another decomposition  $\pi = g' \circ h'$ . Then, by the uniqueness of decomposition of homomorphisms of algebras, we get an isomorphism;  $\alpha: X \rightarrow A/\theta$



**Figure 6:** Two decompositions of  $\pi$

such that,  $\alpha \circ h = h'$  and  $g' \circ \alpha = g$ . ■

## 5. Conclusion and Recommendation

On the basis of the Updated Fundamental Theorem of Homomorphisms of General Universal Algebras which is formulated and proved in this paper, we can conclude that any homomorphism of algebras of a given type can be decomposed as a composition of a monomorphism and an epimorphism and this decomposition is unique upto associate; in the sense that, if a given homomorphism  $\pi$  of algebras of a given type can be decomposed in one way as  $\pi = g \circ h$  where  $g$  is a monomorphism and  $h$  is an epimorphism, and if  $\pi$  can also be decomposed in another way as  $\pi = g' \circ h'$  where  $g'$  is a monomorphism and  $h'$  is an epimorphism then we get that  $g \sim g'$  (or  $g$  and  $g'$  are associate to each other) and  $h \sim h'$  (or  $h$  and  $h'$  are associate to each other). We say that two homomorphisms  $g$  and  $h$  of algebras of a given type are associate to each other, if there exists two isomorphisms  $\alpha$  and  $\beta$  such that;  $g \circ \alpha = \beta \circ h$ . Since  $\alpha$  and  $\beta$  are isomorphisms and hence invertible it is equivalently saying that one can be expressed as a composition of the other and two isomorphisms; that is,  $g \circ \alpha = \beta \circ h \Leftrightarrow g = \beta \circ h \circ \alpha^{-1} \Leftrightarrow h = \beta^{-1} \circ g \circ \alpha$ . It follows from this definition that, two associate homomorphisms have different properties in common such as: one of the two associate homomorphisms is an injection (respectively a surjection and a bijection) if and only if the other is an injection (respectively a surjection and a bijection). Moreover, their domain (respectively codomain) are isomorphic to each other. It is finally recommended that it needs additional researches on the class of associate homomorphisms of algebras of a given type to identify and characterize in a more general way.

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## References

- [1] Burris, S. and Sankappanavor, H.P., *A Course in Universal Algebra*, Springer-Verlag, GTM78, 1981
- [2] Clifford Bergman, *Universal Algebra: Fundamentals and Selected Topics*, CRC Press, 2011
- [3] Gertude Ehrlich, *Fundamental Concept of Abstract Algebra*, Courier Corporation, 2013
- [4] Jacobson, N., *Basic Algebra-Vol I*, W. H. Freeman, San Francisco, 1980
- [5] Jonathan K. Hodge, Steven Schlicker, Ted Sundstrom, *Abstract Algebra: An Inquiry Based Approach*, CRC Press, 2013
- [6] Klaus Denecke, Shelly L. Wismath, *Universal Algebra and Coalgebra*: World Scientific, 2009
- [7] Klaus Denecke, Shelly L. Wismath, *Universal Algebra and Applications in Theoretical Computer Science*: CRC Press, 2002 [P 47-63]
- [8] L. Szabó, A. Szendrei *Lectures in Universal Algebra, Volume 1 of Colloquia Mathematica Societatis Janos Bolyai*, Elsevier, 2014
- [9] Paul B. Garrett, *Abstract Algebra*, CRC Press, 2007 (page 25-56)
- [10] Peter Hilton, Yel-Chiang Wu, *A Course in Modern Algebra: Pure and applied mathematics Volume 12 of Wiley Classics Library*, John Wiley & Sons, 1989
- [11] Pierce, R.S., *Introduction to the theory of Abstract Algebra*, Halt, Rinehart and winston, New York, 1968